NPS55-78-016

NAVAL POSTGRADUATE SCHOOL

Monterey, California

FIRST ORDER AUTOREGRESSIVE GAMMA SEQUENCES

by

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and

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August 1978

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NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

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This research was supported by the National Science Foundation and the Office of Naval Research.

This report was prepared by:

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SECURITY CLASSIFICATION OF THIS PAGE *(When Data Entered)*

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER NPS55-78-016	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE *(and Subtitle)* First Order Autoregressive Gamma Sequences		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) D. P. Gaver and P. A. W. Lewis		8. CONTRACT OR GRANT NUMBER(s) MCS-77-07587
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, Ca. 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School Monterey, Ca. 93940		12. REPORT DATE August 1978
		13. NUMBER OF PAGES 55
14. MONITORING AGENCY NAME & ADDRESS *(if different from Controlling Office)* National Science Foundation Washington, D.C.		15. SECURITY CLASS. *(of this report)* Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT *(of this Report)*

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT *(of the abstract entered in Block 20, if different from Report)*

18. SUPPLEMENTARY NOTES

19. KEY WORDS *(Continue on reverse side if necessary and identify by block number)*

Point processes, time series, Poisson process, Markov process, simulation

20. ABSTRACT *(Continue on reverse side if necessary and identify by block number)*

An autoregressive model that generates Markov correlated time series is described. The time series have exponential or gamma distributed marginal distributions. Various properties of these time series are investigated.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-014-6601

SECURITY CLASSIFICATION OF THIS PAGE *(When Data Entered)*

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FIRST ORDER AUTOREGRESSIVE GAMMA SEQUENCES AND POINT PROCESSES

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1. INTRODUCTION

The Poisson process is a basic model for point processes (series of events) and can be characterized as a process in which the intervals between events are independent and exponentially distributed. In one of the earliest papers on point processes (Wold, 1948) an attempt was made to generalize the Poisson process by obtaining dependent but marginally exponentially distributed intervals between events. Similarly Cox (1955) attempted to obtain a sequence of random variables with conditionally exponential distributions. Neither of these attempts to generalize the Poisson process led to analytically tractable results. In principle, of course, it is simple to generate a sequence of marginally exponentially distributed random variables with Markov dependence if a bivariate exponential random variable is available. However, despite the recent discovery of many bivariate exponential random

*Research of both authors was supported in part by NSF grant MCS-77-07587 at the Naval Postgraduate School and by the Office of Naval Research.

variable generation schemes [Marshall and Olkin (1967), Hawkes (1972); Gaver (1972); Downton (1970)], none seem to have simple enough properties, conditional and otherwise, to lead to analytically and computationally tractable models for a non-independent (Markovian) and easily simulated sequence of marginally exponentially distributed random variables. Interest in the aforementioned processes also may have been damped by the development of physically motivated point process models such as the Poisson cluster process and the self-exciting process, for example see Neyman and Scott (1972). Unfortunately, many of these point process models are somewhat awkward to handle analytically or computationally.

In this paper we show that if one starts with the usual linear, additive autoregressive equation (first-order stochastic difference equation)

$$X_n = \rho X_{n-1} + \varepsilon_n , \qquad n = 0, \pm 1, \pm 2, \ldots \qquad (1.1)$$

where the innovation sequence $\{\varepsilon_n\}$ is one of i.i.d. random variables, then there is, for $0 \leq \rho < 1$, a distribution for the ε_n's such that the X_n's have, marginally, an exponential distribution. The resulting exponential autoregressive process (EAR1) has several attractive features. First, the sequence is obtained as an additive random linear combination of random variables, and is easy to simulate, i.e. to realize on a computer. In simulating a queue, for instance, one can easily obtain sequences of correlated service times or interarrival times, with respectively an i.i.d. exponential sequence or Poisson process as a special case $(\rho = 0)$. These correlated sequences are useful for checking for the sensitivity of standard queueing results to departures from independence, and for other robustness studies.

A second feature is that the EAR1 process (1.1) is analytically tractable; one can, for instance, obtain the Laplace transform of the distribution of sums of the random variables. This is essential if the process is to be used as a model for point processes and one is to obtain the point spectrum.

A third feature of the EAR1 process is that its structure leads to an extended model for a sequence of marginally exponentially distributed random variables with the correlation structure of a mixed autoregressive moving average process (ARMA p,q), the orders of the autoregression and the moving average being p and q respectively. Various properties of this exponential process, called the EARMA(p,q) process, are detailed in Lawrance and Lewis (1977), Jacobs and Lewis (1977) and Lawrance and Lewis (1978).

In Section 2 of this paper we discuss the exponential solution of equation (1.1), as well as questions of stationarity and mixing. The distribution of sums of random variables from this process is discussed next; this relates in particular to the use of the process to model intervals between events in point processes. Joint distributions and conditional correlations of two and three random variables are derived in Section 4. The estimation of the exponential parameter λ, the reciprocal of $E(X_n)$, and the correlation parameter ρ are briefly considered in Section 5 for a fixed or random number of observed random variables. It is shown that, because of a certain degeneracy in the process, ρ can be estimated exactly in a long enough sequence.

Other solutions to equation (1.1) are considered in Section 6. In fact random variables for which solutions to (1.1) (in transformed version) exist are known as self-decomposable random variables or random variables of type L; see Feller (1971, p. 588-90). Two cases of interest in modelling point processes are discussed. These require consideration of Gamma-distributed intervals, and intervals with a mixed exponential distribution; questions connected with the latter remain unresolved.

Finally in Section 7 we consider the question of obtaining autoregressive and Markovian exponential sequences with negative correlation. It may be seen that (1.1) has no solution if ρ is negative. However, as soon as multivariate exponential sequences are considered--in particular, multivariate sequences which are antithetic realizations of each other --we discover a method for obtaining processes with exponential or other specified marginals and that exhibit negative correlation.

2. EXPONENTIAL SEQUENCES

In what follows $\{E_n\}$ will always be a sequence of i.i.d. exponentially distributed random variables with parameter λ:

$$P\{E_n \leq x\} = F_{E_n}(x) = 1 - e^{-\lambda x} \qquad (x \geq 0, \quad \lambda > 0)$$
$$= 0 \qquad (x < 0) \; . \tag{2.1}$$

Also an exponential (λ) random variable will mean a random variable with distribution (2.1).

2.1. <u>The exponential first-order autoregressive sequence (EAR1)</u>.

The starting point of the work in this paper is the question as to whether the autoregressive equation (1.7)

$$X_n = \rho X_{n-1} + \varepsilon_n = \sum_{j=0}^{\infty} \rho^j \varepsilon_{n-j} \qquad |\rho| < 1 \tag{2.2}$$

has a solution for a given distribution of X_i (note that the expansion in (2.2) as an infinite moving average is valid because $|\rho| < 1$). Now X_{n-1} is a function only of $\varepsilon_{n-1}, \varepsilon_{n-2}, \ldots$ and is therefore independent of ε_n. Therefore the Laplace-Stieltjes transform $\phi_{X_n}(s)$ of the distribution of X_n is, at least for $s > 0$,

$$\phi_{X_n}(s) = E[\exp(-sX_n)] = E[\exp(-s\rho X_{n-1} + \varepsilon_n)]$$
$$= \phi_{X_{n-1}}(\rho s) \; \phi_{\varepsilon_n}(s) \; .$$

Thus

$$\phi_{\varepsilon_n}(s) = \frac{\phi_{X_n}(s)}{\phi_{X_n}(\rho s)} \tag{2.3}$$

Assuming that the X_n sequence is marginally stationary, we get the basic equation

$$\phi_{\varepsilon}(s) = \frac{\phi_X(s)}{\phi_X(\rho s)}$$

Now it is clear that if we require the X_n to be positive random variables, then if ρ is negative so is ρX_n and we need the error term ε_n, which is independent of X_{n-1}, to make X_n positive. Thus for positive random variables there will clearly be no solution to (2.3) for $\rho < 0$. However if we let $0 \leq \rho < 1$ and, side-stepping the general question of existence and uniqueness of solutions to (2.3), require the X_n's to be exponential with

$$\phi_X(s) = \frac{\lambda}{\lambda + s},$$

then equation (2.3) forces

$$\phi_{\varepsilon}(s) = \frac{\lambda}{\lambda + s} \cdot \frac{\lambda + \rho s}{\lambda} = \frac{\lambda + \rho s}{\lambda + s} \qquad (0 \leq \rho < 1,\ \lambda \geq 0)$$

$$= \rho + (1-\rho)\frac{\lambda}{\lambda + s}. \tag{2.4}$$

The latter expression is the Laplace-Stieltjes transform of a random variable, in fact a non-negative random variable which has an atom of mass ρ at zero and which is exponential(λ) if positive. Thus we can write the difference equation generating the series $\{X_n\}$ as

$$X_n = \rho X_{n-1} + \varepsilon_n$$

$$= \begin{cases} \rho X_{n-1} & \text{w.p. } \rho \\ \rho X_{n-1} + E_n & \text{w.p. } (1-\rho) \end{cases} \qquad 0 \leq \rho < 1 \tag{2.5}$$

$$= \rho X_{n-1} + I_n E_n, \tag{2.6}$$

where $\{I_n\}$ is an i.i.d. sequence in which $I_n = 1$ with probability $(1-\rho)$, $I_n = 0$ with probability ρ and $\{E_n\}$ is an i.i.d. sequence of (λ) exponentials. It is easily verified, directly from (2.2) or from the definition of ε_n, that

$$E(\varepsilon_n) = (1-\rho)\frac{1}{\lambda} \quad ; \qquad \mathrm{var}(\varepsilon_n) = \frac{1}{\lambda^2}(1-\rho^2) \ . \tag{2.7}$$

There are several points to be made about the sequence X_n:

(i) When $\rho = 0$, then $\{X_n\}$ is an i.i.d. sequence of exponential(λ) r.v.s.

(ii) The representations (2.5) and (2.6) of the process as an (additive) random linear combination of X_{n-1} and E_n, which has a distribution independent of ρ, are sometimes more convenient than is (2.2).

(iii) Although X_n is strictly a linear process (AR1), it differs from the normal or Gaussian process ((2.2) with ε_n normally distributed of Gaussian) in that <u>the mean, variance and higher moments of the</u> ε_n<u>'s are functions</u>

of the parameter ρ. Thus one cannot apply general theorems about linear processes to the exponential sequence because such theorems usually assume that the ε_n's have a distribution which is free of the parameter ρ. As an example, X_n is exponentially distributed even when ρ is close to one, despite theorems for linear processes which assert that in this case the X_n are approximately normally distributed. For instance, if the ε_n's are independent of ρ then the skewness of the stationary distribution of X_n can be shown to approach zero as $\rho \uparrow 1$; in the special case of (2.4), this quantity is independent of ρ and equal to 2--the value associated with the exponential d.f.

(iv) There are many other sequences of random variables $\{X_n\}$ with exponentially distributed marginals and the first-order Markov property. In fact, given any bivariate exponential distribution $F_{E_1,E_2}(x_1,x_2)$, assumed for convenience to be absolutely continuous, with conditional density $f_{E_2|E_1}(x_2;x_1)$, then we construct a Markovian sequence with joint density for, say, X_n, X_{n-1}, ... , X_0 as

$$f_{X_n,\dots,X_0}(x_n,\dots,x_0)$$
$$= f_{E_n|E_{n-1}}(x_n;\ x_{n-1})\ f_{E_{n-1}|E_{n-2}}(x_{n-1};\ x_{n-2})$$
$$\cdots f_{E_1|E_0}(x_1;x_0)\ f_{E_0}(x_0) \quad .$$

An interesting and useful feature of the EAR1 sequence given by the solution (2.6) of the autoregressive equation is that it is a (random) <u>linear</u> combination of i.i.d. exponential sequences, and is thus easy to simulate on a computer, gives reasonably tractable analytic results and in turn suggests the definition of processes with more complicated correlation structure and exponential marginals (Lawrance and Lewis, 1977, Jacobs and Lewis, 1977, Lawrance and Lewis, 1978). The correlation structures are essentially the same as those for linear processes (Lawrance and Lewis, 1978).

2.2. Serial correlations, stationarity and mixing

Simple computations show that, as is true of any regular Markov process, the serial correlations for the EAR1 process are

$$c_j = \mathrm{corr}(X_n X_{n+j}) = \rho^j , \qquad 0 \leq \rho < 1.$$

Note that the correlations are always positive. The spectrum of the sequence is

$$f_+(\omega) = \frac{1}{\pi} \{1 + 2 \sum_{j=1}^{\infty} c_j \cos(j\omega)\} \qquad (0 \leq \omega \leq \pi;\ 0 \leq \rho < 1)$$

$$= \frac{1}{\pi} \quad \frac{1 - \rho^2}{1 + \rho^2 - 2\rho \cos \omega} .$$

For $\rho = 0$ we have $f_+(\omega) = 1/\pi$ since the sequence X_n is i.i.d.

Moreover it is not difficult to show that the process is stationary if one takes

$$X_0 = E_0 \tag{2.8}$$

$$X_n = \rho X_{n-1} + \varepsilon_n = \rho X_{n-1} + I_n E_n \tag{2.9}$$

with ε_n defined by (2.4). This is no surprise, for the process is Markovian and is constructed to have an exponential distribution in the stationary case.

With $E_0 = X_0$ it is shown in Jacobs and Lewis (1977) that the sequence $\{X_n\}$ is strong mixing in the sense of Rosenblatt (1971).

3. SUMS OF STATIONARY INTERVALS

An important aspect of the stationary sequence $\{X_n\}$, especially when it is used to model the intervals in a point process, is the moments and distributions of sums $T_r = X_n + X_{n+1} + \cdots + X_{n+r-1}$. The Laplace-Stieltjes transform of T_r is found directly by noting that, in terms of X_n, (2.2) may be modified to give this representation:

$$X_{n+j} = \rho^j X_n + \rho^{j-1}\varepsilon_{n+1} + \rho^{j-2}\varepsilon_{n+2} + \cdots + \varepsilon_{n+j}, \quad (j = 0,1,2,\ldots) \tag{3.1}$$

and so

$$T_r = \sum_{j=0}^{r-1} X_{n+j} = X_n \left(\frac{1-\rho^r}{1-\rho}\right) + \sum_{j=1}^{r-1} \varepsilon_{n+j} \left(\frac{1-\rho^{r-j}}{1-\rho}\right).$$

It follows that

$$\phi_{T_r}(s) = E[\exp(-sT_t)] = \phi_{X_n}\left[s\left(\frac{1-\rho^r}{1-\rho}\right)\right] \prod_{j=1}^{r-1} \phi_\varepsilon\left[s\left(\frac{1-\rho^{r-j}}{1-\rho}\right)\right] \tag{3.2}$$

If the process starts with the stationary distribution, i.e., if X_n is exponential(λ), then

$$\phi_{T_r}(s) = \frac{\lambda}{\lambda + s\left(\frac{1-\rho^r}{1-\rho}\right)} \prod_{j=1}^{r-1} \left(\frac{\lambda + s\rho\left(\frac{1-\rho^{r-j}}{1-\rho}\right)}{\lambda + s\left(\frac{1-\rho^{r-j}}{1-\rho}\right)}\right) \tag{3.3}$$

Clearly the distribution of T_r can be found explicitly by inverting (3.3), which can be accomplished by expanding in partial fractions. So also can an expression for the interval spectrum. The result is analytically awkward, and will not be quoted. In order to obtain (i) the intensity function, $m_f(t)$, of a point process with EAR1 intervals, (ii) the point spectrum $g_+(\omega)$, and (iii) the distribution of the counting process $N(t)$ (for definitions see Cox and Lewis, 1966, Sec. 4.5) one must be able to sum $\phi_{T_r}(s)$ over r; there appears to be no neat explicit formula for any of these functions. The variance of T_r may be calculated directly from (3.1) since the innovations $\{\varepsilon_n\}$ are independent:

$$\mathrm{Var}[T_r] = \frac{1}{\lambda^2}\left[\left(\frac{1-\rho^r}{1-\rho}\right)^2 + \left(\frac{1+\rho}{1-\rho}\right)\left\{r - 1 - 2\rho\left(\frac{1-\rho^{r-1}}{1-\rho}\right) + \rho^2\left(\frac{1-\rho^{2(r-1)}}{1-\rho^2}\right)\right\}\right] \tag{3.4}$$

Hence for $\rho < 1$ the index of dispersion (Cox and Lewis (1966), p. 71) is

$$J = \lim_{r \to \infty} J_r = \lim_{r \to \infty} \frac{\mathrm{Var}[T_r]}{r\{E[X_1]\}^2} = \frac{1+\rho}{1-\rho} = 1 + \frac{2\rho}{1-\rho} \tag{3.5}$$

As a byproduct of (3.5) we get, for $\rho < 1$, values of the slope of the variance time curve, $\lim_{t \to \infty} V'(t) = V'(\infty)$, the initial points of the spectrum of counts $g_+(\omega)$ and the spectrum of intervals as (Cox and Lewis, 1966, Sect. 4.6)

$$f_+(0+) = \frac{J}{\pi}$$

$$V'(\infty) = \lambda J$$

$$g_+(0+) = \lambda J/\pi$$

Limit theorems for T_r and the counting process $N(t)$ are given in Jacobs and Lewis (1977).

4. JOINT DISTRIBUTIONS FOR THE STATIONARY SEQUENCE

Any pair X_n, X_{n+r} in the stationary EAR1 sequence has a bivariate exponential distribution. Consider X_n and X_{n+1}. Directly from the definition (2.5) and (2.7) it can be shown that the conditional random variable X_{n+1}, given $X_n = x$, has an atom of mass ρ at ρx; otherwise it is, with probability $(1-\rho)$, an exponential(λ) random variable shifted ρx from the origin. The regression of X_{n+1} on $X_n = x$ is

$$E(X_{n+1}|X_n = x) = x + (1-\rho)\frac{1}{\lambda}, \tag{4.1}$$

which is linear. Moreover $\mathrm{var}(X_{n+1}|X_n = x) = (1-\rho^2)/\lambda^2$, a constant independent of x, and X_{n+1} is never smaller than ρX_n. The Laplace-Stieltjes transform of the bivariate exponential distribution of X_{n+1} and X_n is

$$\begin{aligned}
\phi_{X_n;X_{n+1}}(s_1,s_2) &= E\{\exp(-s_1X_n - s_2X_{n+1})\} \\
&= E\{\exp[-s_1 + \rho s_2)X_n]\}\ \phi_{\varepsilon_n}(s_2) \\
&= \frac{\lambda}{\lambda + s_1 + \rho s_2}\left\{\rho + (1-\rho)\ \frac{\lambda}{\lambda + s_2}\right\} \\
&= \rho\,\frac{\lambda}{\lambda + s_1 + \rho s_2} + (1-\rho)\left(\frac{\lambda}{\lambda + s_1 + \rho s_2}\right)\left(\frac{\lambda}{\lambda + s_2}\right)
\end{aligned} \tag{4.2}$$

The bivariate distribution of $\{X_{n+1}, X_n\}$ has a singular component along the line $X_{n+1} = X_n$, with probability ρ (the first term in (4.2), and a continuous component in the space $X_{n+1} > \rho X_n$ (the second term in (4.2)) which is the joint distribution of X_n and $\rho X_n + E_n'$. Note that the distribution is not symmetric in X_{n+1} and X_n, since the transform $\phi_{X_{n+1},X_n}(s_1,s_2)$ is not symmetric in s_1 and s_2. This simply means that the process is not time-reversible, as is the normal or Gaussian AR1 process.

This asymmetry appears in the conditional mean of X_n, given $X_{n+1} = x$, which is obtained from (4.2), as

This asymmetry also becomes evident in the non-linear form of the conditional mean of X_n, given $X_{n+1} = x$, which is obtained from (4.2), as

$$E(X_n | X_{n+1}=x) = \frac{1}{\lambda(1-\rho)} \{1 - \exp[-(\frac{1}{\rho} - 1)\lambda x]\} \neq E[X_{n+1} | X_n=x] .$$

The conditional variance of X_n, given $X_{n+1} = x$, can also be obtained; unlike $\mathrm{var}(X_{n+1} | X_n = x)$ it is not a constant.

Higher order directional moments of X_{n+1}, X_n are given in Jacobs and Lewis (1977), and again evidence the directionality of the process:

$$C_{2,1}(1) = E(X_n^2 X_{n+1}) - E(X_n^2) E(X_{n+1}) = \frac{4\rho}{\lambda^3} ;$$

$$C_{1,2}(1) = E(X_n X_{n+1}^2) - E(X_n) E(X_{n+1}^2) = \frac{2\rho(1+\rho)}{\lambda^3} .$$

Since the process is, from its definition, Markovian, the conditional correlation of X_{n+1} and X_{n-1}, given $X_n = x$, is zero. We do not discuss distributions of triples any further. In principle it is simple to write down transforms of the joint distribution of any set of k X_n's; one obvious use of this result would be in obtaining the distribution of the sum of $X_n, \ldots, X_{n+k-1}$. However, this was obtained by a direct argument in the previous section.

5. ESTIMATION AND DEGENERACY

Although it is not possible to write down the likelihood equation for the EAR1 process in a tractable form where, say, we observe $X_1, \ldots, X_N$, it seems at first sight that good estimates of $1/\lambda$ and ρ can be obtained from the sample mean, $\bar{X} = T_n/n$, whose variance is given at (3.4), and a first-order serial correlation coefficient estimate $\hat{\rho}_1$ with modifications to take account of the fact that the variance equals the mean squared, i.e. $\mathrm{Var}(X) = 1/\lambda^2$.

However, there is a degeneracy in the process which makes it possible to find ρ exactly in a long enough run of X_n's, and to estimate λ more precisely than is possible using straightforward moment methods.

To identify the degeneracy, which we call the <u>zero-defect</u>, we note that in the process there are runs of X_n's which are equal to the previous value, X_{n-1}, times ρ. Moreover, given that there is a run of length R of this type, R has a "geometric plus one" distribution with parameter ρ;

$$P\{R = i\} = (1 - \rho)\rho^{i-1} , \quad i = 1,2,3, \ldots \tag{5.1}$$

$$E(R) = \frac{\rho}{1 - \rho} ; \quad \mathrm{var}(R) = \frac{\rho}{(1 - \rho)^2} .$$

This behavior, or a tendency towards it in data would be very evident in plots of a sample sequence of X_n values.

Thus to estimate ρ in an observed series, let $Z_n = X_{n+1}/X_n$, for $n = 2,3, \ldots$. Then

$$Z_n = \frac{X_{n+1}}{X_n} = \rho \qquad \text{if } X_{n+1} = \rho X_n \quad \text{w.p. } \rho$$

$$= \rho + \frac{E_{n+1}}{X_n} \qquad \text{if } X_{n+1} \neq \rho X_n \quad \text{w.p. } (1-\rho) .$$

The probability that $Z_n = \rho$, or $Z_n \neq \rho$, is independent of previous Z_n's and $P\{Z_n = \rho\} = \rho$. Hence if we observe the minimum value of Z_n, $n = 2,3,\ldots$ until we get a tie, that tie value is ρ. The time to obtain this tie is the sum of two "geometric plus one" random variables, with total mean $2/\rho$ and total variance $2(1-\rho)/\rho^2$. The implication is that we can find ρ exactly after a random number of observations for the present model. Moreover since one then knows those X_n's which are made up of ρX_{n-1} plus an E_n, it is possible to compute the E_n's exactly (except for two) and estimate $E(X) = 1/\lambda$ directly from the observed E_n's.

For a fixed length observation of the process, say $X_1, \ldots, X_N$, the probability that ρ can be obtained exactly is the probability that the sum of two "geometric plus one" random variables is less than or equal to N. If two minimum values are not observed by time N, a good estimate of ρ will be the minimum of the Z_i's. This will either be ρ, or ρ plus a small bias term. The mean can be estimated as the sample average in the usual way.

This degeneracy in sequences of positive random variables generated from the stochastic difference equation seems to be inherent in the present procedure. Slightly more realistic but complicated first-order autoregressive exponential processes without this zero defect will be discussed elsewhere.

6. OTHER SOLUTIONS TO THE AUTOREGRESSIVE EQUATION

Random variables X for which the transformed first-order autoregressive equation

$$\frac{\phi_X(s)}{\phi_X(\rho s)} = \phi_\varepsilon(s) \tag{6.1}$$

has a solution $\phi_\varepsilon(s)$ for each $0 < \rho < 1$ which is the transform of a distribution function are called <u>random variables of class L</u> (Feller, 1971, p. 588), <u>or self-decomposable random variables</u>. Although there has been much recent interest in finding class L random variables, the connection with the first-order autoregressive process does not seem to have been made. On the other hand there have been some attempts to find non-normal solutions to the autoregressive equation without connecting them to class L theory (see, e.g. Bernier, 1970).

The limitation of the theory of class L random variables as it relates to the present work is that it requires a solution for each $0 < \rho < 1$, which is the case for exponential(λ) random variables X. This full range of ρ is desirable, but may not occur. We do not explore the full connection here, but consider only two types of random variables, Gamma distributed random variables and mixed exponential random variables. These are frequently used as alternatives to the exponential in modelling stochastic phenomena such as response times in queues.

6.1. The Gamma Autoregressive Process GAR1.

A necessary condition for X to be in the class L is that it be infinitely divisible, and therefore we consider Gamma (λ,k) variables with density

$$f_X(x) = \frac{\lambda^k x^{k-1} e^{-}}{k^k \Gamma(k)} \qquad (\lambda \geq 0;\ k > 0;\ x \geq 0) \tag{6.2}$$

and Laplace-Stieltjes transform (of the stationary distribution)

$$\phi_X(s) = \left(\frac{\lambda}{\lambda + s}\right)^k . \tag{6.3}$$

Then the solution to (6.1) is

$$\phi_\varepsilon(s) = \left(\frac{\lambda + \rho s}{\lambda + s}\right)^k = \left(\rho + (1-\rho)\frac{\lambda}{\lambda + s}\right)^k \tag{6.4}$$

It follows from a result reported by Feller ((1971), p. 452) or directly from Theorem 1, p. 450) that $[(\lambda + \rho s)/(\lambda + s)]$ is the transform of an infinitely divisible distribution, and hence (6.3) is actually the transform of (an infinitely divisible) distribution function for every real $k > 0$. Thus we can in principle generate an autoregressive process with gamma marginals (6.2) by utilizing the $\{\varepsilon_n\}$ process characterized by (6.3). Here are a few simple special cases.

$$k = 2: \quad \phi_\varepsilon(s) = \rho^2 + 2\rho(1-\rho)\frac{\lambda}{s + \lambda} + (1-\rho)^2\left(\frac{\lambda}{s + \lambda}\right)^2 ; \tag{6.5}$$

$$k = 3: \quad \phi_\varepsilon(s) = \rho^3 + 2\rho^2(1-\rho)\frac{\lambda}{s + \lambda} + 2\rho(1-\rho)^2\left(\frac{\lambda}{s + \lambda}\right)^2 + (1-\rho)^3\left(\frac{\lambda}{s + \lambda}\right)^3 . \tag{6.6}$$

Thus for $k = 2$ the random variable ε may be thought of and realized as a convex mixture of a degenerate random variable with mass at zero and an exponential(λ) and a Gamma (λ,2) distribution. Alternatively if E_1 and E_2 are independent exponential(λ) random variables, the sum is chosen with probability $(1-\rho)^2$, E_1 is chosen with probability $2\rho(1-\rho)^2$, and neither of E_1 or E_2, but rather zero, is chosen with probability ρ^2.

Unfortunately, no really simple way of generating random variables with Laplace-Stieltjes transform (6.4) for noninteger k is known, though representation as an infinite sum of weighted Gamma variables is possible (Bernier (1970)). Moreover for a case of much interest $k < 1$, so that X is overdispersed relative to an exponential random variable, and the degeneracy or zero-defect of weight ρ^k becomes very prominent.

As was true for the exponential case, we have for the correlations, which are all positive,

$$c_j = \rho^j \qquad 0 \leq \rho < 1;$$

also

$$E(\varepsilon) = \lambda(1-\rho)$$

$$\mathrm{Var}(\varepsilon) = \lambda^2(1-\rho^2)/k \; .$$

As $k \to \infty$, the degeneracy disappears and X tends to become normally distributed.

6.2. The Mixed Exponential Process MEAR1

Consider now X to be a mixture of two exponential densities

$$f_X(x) = \pi_1\lambda_1 e^{-\lambda_1 x} + \pi_2\lambda_2 e^{-\lambda_2 x} ,$$

$$x \geq 0, \quad \pi_1 = 1 - \pi_2 \geq 0, \quad \lambda_1 < \lambda_2 . \tag{6.7}$$

For $\pi_1 = 0$, $\pi_1 = 1$, and/or $\lambda_1 = \lambda_2$ this of course reduces to an exponential density. There are two main cases to consider, besides these special cases.

If $0 \leq \pi_1 \leq 1$, then we always have a convex mixture of exponential densities: $f_X(x)$ is a proper density function and the coefficient of variation of X lies between 1 and infinity (Cox, 1964), so that X is overdispersed relative to an exponential random variable; X may be generated as a mixture of two exponential random variables with parameters λ_1 and λ_2, and X is infinitely divisible (Feller (1971), p. 452).

If π_1 is greater than 1, so that π_2 is negative, a necessary and sufficient condition for $f_X(x)$ to be a p.d.f. is that $\pi_1\lambda_1 + \pi_2\lambda_2 \geq 0$ or equivalently that $\pi_1 \leq [1 - (\lambda_1/\lambda_2)]^{-1}$.

Here is alternative way of writing the transform of X. First, and directly,

$$\phi_X(s) = \pi_1 \frac{\lambda_1}{\lambda_1 + s} + \pi_2 \frac{\lambda_2}{\lambda_2 + s} \tag{6.8}$$

Now since $\lambda_2 > \lambda_1$ factor to obtain

$$\phi_X(s) = \frac{\lambda_2}{\lambda_2 + s}\left\{\pi_2 + \pi_1 \frac{\lambda_1}{\lambda_2}\left(\frac{\lambda_2 + s}{\lambda_1 + s}\right)\right\}$$

$$= \frac{\lambda_2}{\lambda_2 + s}\left\{1 - \pi_1\left(1 - \frac{\lambda_1}{\lambda_2}\right) + \pi_1\left(1 - \frac{\lambda_1}{\lambda_2}\right)\frac{\lambda_1}{\lambda_1 + s}\right\}. \qquad (6.9)$$

Now even if $\pi_1 > 1$ $(\pi_2 < 0)$ but

$$\gamma = \pi_1\left(1 - \frac{\lambda_1}{\lambda_2}\right) < 1 , \qquad (6.10)$$

or equivalently

$$\pi_2\lambda_2 + \pi_1\lambda_1 > 0 \qquad (6.11)$$

then $\phi_X(s)$ is the transform of the sum of an exponential(λ_2) random variable and a random variable having the distribution of the ε-innovation of an EAR1 process, see (2.4). It follows that X is infinitely divisible.

It is thus worth investigating whether one can generate an autoregressive process $\{X_n\}$ with mixed exponential marginals for some or all ρ, the latter being the question as to whether the mixed exponential random variable is in the class L. We have, directly

$$\phi_\varepsilon(s) = \frac{\phi_X(s)}{\phi_X(\rho s)} = \frac{\dfrac{\pi_1\lambda_1}{s+\lambda_1} + \dfrac{\pi_2\lambda_2}{s+\lambda_2}}{\dfrac{\pi\lambda_1}{\rho s+\lambda_1} + \dfrac{\pi_2\lambda_2}{\rho s+\lambda_2}}$$

$$= \frac{(\lambda_1+\rho s)(\lambda_2+\rho s)(\lambda_1\lambda_2+\pi_1\lambda_1 s+\pi_2\lambda_2 s)}{(\lambda_1+s)(\lambda_2+s)(\lambda_1\lambda_2+\pi_1\lambda_1\rho s+\pi_2\lambda_2\rho s)} \tag{6.12}$$

After considerable simplification this Laplace-Stieltjes transform can be written as a mixture of a degenerate random variable with mass ρ at zero, and with probability $(1-\rho)$ (possibly) a random variable Y which has (possibly) density function

$$f_Y(x) = \gamma_1\lambda_1 e^{-\lambda_1 x} + \gamma_2\lambda_2 e^{-\lambda_2 x} + \gamma_3 \frac{C}{\rho} e^{-xc/\rho} \tag{6.13}$$

where $\gamma_1+\gamma_2+\gamma_3 = 1$, $\lambda_1 < \lambda_2$, $\pi_1 > 0$, $\pi_1 \neq 1$ and

$$C = \frac{\lambda_1\lambda_2}{\pi_1\lambda_1 + (1-\pi_1)\lambda_2};$$

$$\gamma_1 = \frac{(\lambda_2 - \rho\lambda_1)(C - \lambda_1)}{(\lambda_2 - \lambda_1)(C - \rho\lambda_1)} = p_1 ; \tag{6.14}$$

$$\gamma_2 = \frac{(\lambda_2 - C)}{(\lambda_2 - \lambda_1)} \times \frac{(\rho\lambda_2 - \lambda_1)}{(\rho\lambda_2 - C)} = p_2 \frac{(\lambda_1 - \rho\lambda_2)}{(C - \rho\lambda_2)}; \quad (C \neq \rho\lambda_2) \tag{6.15}$$

$$\gamma_3 = \frac{(C - \lambda_1)(\lambda_2 - C)}{(C - \rho\lambda_1)} \times \frac{1}{(C - \rho\lambda_2)} = p_3 \times \frac{1}{(C - \rho\lambda_2)} ; \tag{6.16}$$

Now if π_1 is restricted to be $0 < \pi_1 < 1$, then $\lambda_1 < C < \lambda_2$ and $p_1 > 0$, $p_2 > 0$, $p_3 > 0$. Then it can be shown that the signs of γ_1, γ_2, γ_3 depend on whether $0 < \rho \leq \lambda_1/\lambda_2$, $\lambda_1/\lambda_2 < \rho < C/\lambda_2$ or $C/\lambda_2 < \rho < 1$:

(i) γ_1 is always positive;

(ii) γ_2 and γ_3 are positive if $0 < \rho \leq \lambda_1/\lambda_2$;

(iii) γ_2 is negative, γ_3 is positive if $\lambda_1/\lambda_2 < \rho < C/\lambda_1$;

(iv) γ_2 is positive, γ_3 is negative if $C/\lambda_2 < \rho < 1$.

While this result is relatively simple, the only case in which it has been possible to establish that $f_Y(x)$ is a p.d.f. is that in (ii), i.e. $0 < \rho \leq \lambda_1/\lambda_2$. The ratio λ_1/λ_2 is roughly related to the overdispersion of X relative to the exponential distribution; the smaller the value λ_1/λ_2, the greater the dispersion and the smaller the admissible range of ρ. It seems probable that $f_Y(x)$ is not a density function for all ρ; at all events efforts to prove that it is in the class L using (6.10) and characterization theorems (Feller, 1971) have failed at the time of this writing. A complete understanding is not now available.

6.3. Discussion

Note the differences between the mixed exponential autoregressive sequence and the Gamma ($k < 1$) autoregressive sequence; the former appears to require a restricted range for ρ, ε can be simulated, and the zero defect has probability ρ; the latter sequence is valid for $0 \leq \rho < 1$, ε cannot at present be simulated, and the zero-defect has probability $\rho^k > \rho$. The magnitude of this zero-defect alone probably makes the Gamma process less useful than the mixed exponential process. The tail behavior of the marginal distributions is also different.

Laplace-Stieltjes transforms of distribution of sums, T_{r_1} of X_n's are obtained from (3.2) for both the Gamma and mixed exponential autoregressive sequences. These cases are still under study.

7. MULTIVARIATE SEQUENCES AND NEGATIVE CORRELATION

Even though the autoregressive equation (1.1) has no solution representing a positive random sequence when $\rho < 0$ it is possible to modify it so as to produce negatively correlated exponential sequences. These will not be individually Markovian, but will have Markovian properties in a bivariate sense. A suggestion as to how to proceed comes from writing X_n in terms of the innovations $\{\varepsilon_n\}$ as in (2.2):

$$X_n = \sum_{j=0}^{n} \rho^j \varepsilon_{n-j} = \varepsilon_n + \rho\varepsilon_{n-1} + \rho^2\varepsilon_{n-2} + \cdots + \rho_n \varepsilon_0 \, . \tag{7.1}$$

Now consider generating two (or more) series in parallel, using independent pairs of innovations, $(\varepsilon_n, \varepsilon_n')$, for which if desired $\mathrm{Cor}(\varepsilon_n, \varepsilon_n') < 0$; e.g. for $0 \leq \rho < 1$ and starting with the first innovation.

Model 1:
$$\begin{aligned} X_n &= \rho X_{n-1}' + \varepsilon_n = \varepsilon_n + \rho\varepsilon_{n-1}' + \rho^2\varepsilon_{n-2} + \rho^3\varepsilon_{n-3}' + \cdots \\ X_n' &= \rho X_{n-1} + \varepsilon_n' = \varepsilon_n' + \rho\varepsilon_{n-1} + \rho^2\varepsilon_{n-2}' + \rho^3\varepsilon_{n-3} + \cdots \end{aligned} \tag{7.2}$$

Model 2:
$$\begin{aligned} X_n &= \rho X_{n-1} + \varepsilon_n = \varepsilon_n + \rho\varepsilon_{n-1} + \rho^2\varepsilon_{n-2} + \rho^3\varepsilon_{n-3} + \cdots \\ X_n' &= \rho X_{n-1}' + \varepsilon_n' = \varepsilon_n' + \rho\varepsilon_{n-1}' + \rho^2\varepsilon_{n-2}' + \rho^3\varepsilon_{n-3}' + \cdots \end{aligned} \tag{7.30}$$

In order for $\{X_n\}$ and $\{X'_n\}$ to exhibit asymptotically exponential marginal distribution, it is clear that the marginal innovation sequences $\{\varepsilon_n\}$ and $\{\varepsilon'_n\}$ must themselves be iid with an atom ρ at zero, and otherwise an exponential distribution. This can be accomplished in several ways; here are two:

(i) Let

$$\left.\begin{aligned} &\varepsilon_n = \varepsilon'_n = 0 && \text{with probability } \rho,\ 0 \le \rho < 1 \\ &\left.\begin{aligned} \varepsilon_n &= E_n = -\frac{1}{\lambda}\,\ell n(U_n) \\ \varepsilon'_n &= E'_n = -\frac{1}{\lambda}\,\ell n(1-U_n) \end{aligned}\right\} && \text{with probability } 1-\rho \end{aligned}\right. \tag{7.4}$$

where $\{U_n\}$ is a sequence of uniform (0,1) "random numbers"; $\{E_n\}$ and $\{E'_n\}$ are then called antithetic, see Hammersley and Handscomb (1964), and are maximally negatively correlated, having correlation $1 - (\pi^2/6) = -0.6449$; see Moran (1967). Each series in Models 1, 2 receive independent innovations of exactly the type described in Section 2 and that were there shown to lead to exponential marginals as $n \to \infty$. In this case

$$E[\varepsilon,\varepsilon'] = (1-\rho)\,\frac{1}{\lambda^2}\int_0^1 \ell n\, u\ \ell n(1-u)\,du = (1-\rho)\,\frac{1}{\lambda^2}\left[2 - \frac{\pi^2}{6}\right] \tag{7.5}$$

and

$$\mathrm{Cov}(\varepsilon,\varepsilon') = (1-\rho)\,\frac{1}{\lambda^2}\left[2 - \frac{\pi^2}{6}\right] - \frac{(1-\rho)^2}{\lambda^2} \tag{7.6}$$

Also $\mathrm{Cov}(\varepsilon_n,\varepsilon'_n) > 0$ if $\rho > (\pi^2/6 - 1)$.

(ii) If the distribution function of ε_n is F_{ε_n}, then in true antithetic fashion determine

$$\varepsilon_n = F_{\varepsilon}^{-1}(U_n); \qquad \varepsilon_n' = F_{\varepsilon}^{-1}(1-U_n) \tag{7.7}$$

In particular, if

$$F_{\varepsilon_n}(x) = \begin{cases} 0 & \text{if } x < 0 \\ \rho & \text{if } x = 0 \\ \rho + (1-\rho)(1-e^{-\lambda x}) & \text{if } x > 0 \end{cases} \tag{7.8}$$

Then innovation leading to marginal exponentials(λ) are obtained; see Section 2. In this case

$$\begin{aligned} \varepsilon_n &= 0 \quad \text{if } U_n \leq \rho; & \varepsilon_n' &= 0 \quad \text{if } U_n \geq 1-\rho; \\ &= -\frac{1}{\lambda}\,\ell n\left(\frac{1-U_n}{1-\rho}\right) \quad \text{if } U_n > \rho; & &= -\frac{1}{\lambda}\,\ell n\left(\frac{U_n}{1-\rho}\right) \quad \text{if } U_n < 1-\rho \end{aligned} \tag{7.9}$$

Consequently

$$\begin{aligned} E[\varepsilon,\varepsilon_n'] &= \frac{1}{\lambda^2}\int_{\rho}^{1-\rho} \ell n\left(\frac{u}{1-\rho}\right)\ell n\left(\frac{1-u}{1-\rho}\right) du\,, & 0 \leq \rho \leq \tfrac{1}{2} \\ &= 0\,, & \tfrac{1}{2} < \rho \leq 1 \end{aligned} \tag{7.10}$$

and hence

$$\mathrm{Cov}(\varepsilon_n,\varepsilon_n') = \frac{1}{\lambda^2}\int_{\rho}^{1-\rho} \ln\frac{u}{1-\rho}\,\ln\frac{1-u}{1-\rho}\,du - \left(\frac{1-\rho}{\lambda}\right)^2, \qquad 0 \le \rho \le \frac{1}{2}$$

$$= -\left(\frac{1-\rho}{\lambda}\right)^2, \qquad \frac{1}{2} < \rho \le 1 . \tag{7.11}$$

Scheme (ii) is capable of generating negative correlations of greater magnitude than is Scheme (i), as is clear from examining the situation $1/2 < \rho$.

The expansions (7.2) and (7.3) allow the computation of the lagged cross covariances between X_n and X_n'. For $n \to \infty$ we find

$$\text{Model 1:}\quad \mathrm{Cov}(X_{n+j},X_n') \sim \frac{\rho^j}{1-\rho^2}\,\mathrm{Cov}(\varepsilon,\varepsilon'), \quad j = 0,1,2,\ldots \tag{7.12}$$

$$\text{Model 2:}\quad \mathrm{Cov}(X_{n+j},X_n') \sim \frac{\rho^j}{1-\rho^2} \times \begin{cases} \mathrm{Cov}(\varepsilon,\varepsilon') & \text{for } j = 0,2,4,\ldots \\ \mathrm{Var}(\varepsilon) & \text{for } j = 1,3,5,\ldots \end{cases} \tag{7.13}$$

Of interest is the fact that the series of Model 2 may in fact exhibit negative correlations; for, directly from (7.2), and as $n \to \infty$,

$$\mathrm{Cov}(X_{n+1},X_n) = \mathrm{Cov}(X_{n+1}',X_n') \sim \frac{\rho}{1-\rho^2}\,\mathrm{Cov}(\varepsilon,\varepsilon')$$

$$\mathrm{Cov}(X_{n+2},X_n) = \mathrm{Cov}(X_{n+2}',X_n') \sim \frac{\rho^2}{1-\rho^2}\,\mathrm{Var}(\varepsilon)$$

$$\cdots \qquad\qquad \cdots$$

$$\mathrm{Cov}(X_{n+j},X_n) = \mathrm{Cov}(X_{n+j}',X_n') \sim \frac{\rho^j}{1-\rho^2} \times \begin{cases} \mathrm{Cov}(\varepsilon,\varepsilon'), & j = 1,3,5,\ldots \\ \mathrm{Var}(\varepsilon), & j = 2,4,6,\ldots \end{cases}$$

Consequently if scheme (i) or (ii) above, or a multitude of other possibilities, generate the innovation pairs, then the odd-numbered covariances will in fact be negative.

Clearly the above generation scheme can be generalized, e.g. by including more equations, $\{X_n\}$, $\{X'_n\}$, $\{X''_n\}$, etc., and by allowing the innovations to have different marginal properties. Further investigations are planned.

8. SUMMARY AND CONCLUSIONS

We have presented a simple, autoregressive Markovian sequence $\{X_n\}$ of exponential variates EAR1 which is an additive random linear combination of the previous value, X_{n-1}, and an independent exponential random variable. The simplicity of this structure allows one to model in an intuitive way dependencies in stochastic systems. Jacobs (1978) has considered cyclic queues with EAR1 service times and found that the correlation may produce a significant effect; more general queueing schemes which generate multivariate exponential sequences are given by Lewis and Shedler (1978).

Maxima of the X_n in the EAR1 process have been studied by Chernick (1977).

Two other marginal distributions for the X_n's have been studied, the Gamma distribution and the mixed exponential. The former is known to be a type L, i.e. satisfies (6.1) for all ρ, while the mixed exponential appears to satisfy (6.1) only for a limited range of ρ. Other type L distributions will be investigated in the context of the modelling of independent stochastic sequences elsewhere.

Extensions of the first-order autoregressive structure for exponential marginals to higher-order autoregressions, moving averages and mixed autoregressive-moving average structures has been given by Lawrance and Lewis (1977), Jacobs and Lewis (1977) and Lawrance and Lewis (1978). The possibility of extending the EAR1 structure depends on the fact that the $\{\varepsilon_n\}$ error sequence is a mixture of a random variable with mass at zero and an exponential distribution.

Three other possibilities will be detailed elsewhere--a slight complication which brings in another parameter but gets rid of the "zero-deficiency" in the EAR1 process; extension of mixed correlation structures to give negative correlations, and further multivariate extensions.

Finally we note that it is possible to introduce non-stationarity and dependence on concomitant variables into the sequence by multiplying X_n by, say,

$$\lambda(n) = \exp\{\sum_{j=0}^{r} \alpha_j z_j(n)\},$$

as was done in Cox and Lewis (1966, Ch. 3, ii). It would be of interest to see how the methods given in Cox and Lewis extend to the case where there is correlation present between the X_n's.

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